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# Asymptotic analysis of the Ponzano–Regge model for handlebodies

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**Abstract**

Using the coherent state techniques developed for the analysis of the EPRL model we give the asymptotic formula for the Ponzano–Regge model amplitude for non-tardis triangulations of handlebodies in the limit of large boundary spins. The formula produces a sum over all possible immersions of the boundary triangulation and its value is given by the cosine of the Regge action evaluated on these. Furthermore the asymptotic scaling registers the existence of flexible immersions. We verify numerically that this formula approximates the $6j$-symbol for large spins.

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## 1. Introduction

In [1], Ponzano and Regge gave a formula for the large spin limit of the $6j$ symbol. It was found to be related to the Regge action for discrete general relativity and with this motivation they constructed the first spin foam model of 3D gravity. Their asymptotic formula was first proved in [2], then more recently using different methods in [3, 4] and the square of the $6j$ symbol was also studied in the context of relativistic spin networks [5, 6]. The next to leading order approximation was recently considered in [7]. The precise formulation of the full state sum was studied in [8].

In [9, 10], the semiclassical limit of some recent spin foam models [11–13] was analysed using the coherent state techniques introduced in [13]. In particular the boundary there was formulated in terms of coherent tetrahedra. Here we apply the same techniques in the 3D case using coherent triangles and, instead of a single vertex amplitude, we analyse triangulations of arbitrary genus handlebodies. This finally opens up the possibility of a continuum limit and renormalization analysis of the model for this restricted class of 3-manifolds. In particular the resulting formula is well suited for studying the graviton propagator as introduced for Ponzano–Regge in [14].

We begin the paper by describing the formulation of the Ponzano–Regge model in terms of a single spin network diagram dual to the boundary triangulation. We then describe the boundary state choice in detail and proceed to give the asymptotic formula in terms of immersions of the boundary triangulation. The asymptotic scaling of the amplitude has the interesting feature that it registers whether or not there are flexible immersions of the boundary. An explicit example is provided in Steffen's polyhedron. This analysis sheds light on the general way in which asymptotic behaviour for larger triangulations can emerge from a spin foam model. In particular it does not need to proceed by taking the asymptotics of the individual simplex amplitudes first.

## 2. The Ponzano–Regge model in terms of coherent states on the boundary

The Ponzano–Regge amplitude was originally defined in terms of $6j$ symbols with a cutoff regularization on the interior vertices. More recently, it was shown in [8] that the cutoff regularization for sums over representations in some cases disallows a 2–3 Pachner move (the Biedenharn–Elliot identity does not hold for a restricted sum over representations.) This meant topological invariance of the partition function cannot be proved with Pachner moves in this form. The alternative formulation in terms of delta functions and integrals over $SU(2)$ regularized with a gauge fixing tree is both finite and invariant under Pachner moves. Another regularization using representations of a quantum group is given by the Turaev–Viro model; however, it was also shown in [8] that the limiting procedure that gives the Ponzano–Regge model is only known to exist for so-called non-tardis triangulations—i.e. a triangulation whose edge lengths are restricted to a finite range by the boundary edge lengths. In order to avoid discussing regularization, in this paper we will restrict to only considering these non-tardis triangulations which are by definition finite. Slightly extending the terminology of [8], we will call a manifold $\Sigma^3$ a *non-tardis manifold* if there exists a 'non-tardis' triangulation of $\Sigma^3$.

For a 3-manifold $\Sigma^3$ with orientable 2-boundary $\partial\Sigma^3$ its boundary state space is then given by the possible geometric triangulations of the 2-boundary with half-integer edge lengths. The amplitude for such a non-tardis manifold is given in terms of a non-tardis triangulation $\mathcal{T}$ of $\Sigma^3$ that extends the boundary triangulation and some boundary state $\Psi$:

$$\mathcal{Z}_{\mathrm{PR}}(\Psi, \mathcal{T}) = \sum_{j_e} \prod_e \dim(j_e) \prod_\Delta \frac{1}{\langle \mathrm{Theta} \rangle} \prod_\sigma \langle \mathrm{Tet} \rangle. \tag{1}$$

Here $e$ is an edge, $\Delta$ a triangle and $\sigma$ a tetrahedron of the triangulation of the interior, $j$ are half-integers labelling the irreps of $SU(2)$. The amplitudes $\langle\mathrm{Theta}\rangle$ and $\langle\mathrm{Tet}\rangle$ are the spin network evaluation of the theta graph and the planar tetrahedral spin network, respectively. These spin networks are the two-dimensional duals to the interior triangles $\Delta$ and, respectively, to the surface of the tetrahedra $\sigma$ in the triangulation. The labels of the $\langle\mathrm{Theta}\rangle$ and $\langle\mathrm{Tet}\rangle$ networks associated with the triangles and tetrahedra are given by assigning the $j$ associated with each edge to each dual edge that crosses it. Finally $\dim(j) = (-1)^{2j}(2j+1)$ is the dimension of the $j$th $SU(2)$ irrep in graphical calculus given by the evaluation of a single loop diagram in the $j$ representation.

In the interior the normalization and phase of the intertwiners cancels. However, at the boundary these are arbitrary normalization for each face. This information is in the boundary state $\Psi$ which consists of the boundary edge length data and the particular intertwiner chosen at each face on the boundary.

### *2.1. Ponzano–Regge on the boundary*

In some cases it is possible to reformulate the Ponzano–Regge model defined above as a spin network evaluation on the 2-boundary of the manifold. In fact, Ponzano and Regge originally constructed the state sum model such that it agreed with the evaluation of a planar spin network associated with the boundary of a 3-ball. An algorithm to construct a non-tardis interior triangulation given an arbitrary triangulation of the boundary of $B^3$ was given using recoupling theory by Moussouris in [15]. This algorithm consists of reducing the boundary spin network to a product of $6j$ symbols (which is always possible for a planar diagram) using the recoupling identity and Schur's lemma and then reconstructing the interior triangulation from these $6j$ symbols. Since the boundary spin network is finite, this procedure gives a manifestly finite definition of the partition function.

In this paper we will extend this result to spin networks on the boundary of handlebodies of arbitrary genus. A non-tardis triangulation of a handlebody of genus $g$ can be constructed as follows. Start with a triangulation of the boundary of $B^3$ with $g$ distinct pairs of triangles that do not share a common vertex. The boundary of the handlebody can be formed by identifying these triangles and a non-tardis triangulation of the interior is given by applying the Moussouris algorithm. This procedure may result in a degenerate triangulation of the handlebody even if the triangulation of the ball is non-degenerate.

We will begin our analysis by reformulating the amplitude for the 3-ball $B^3$ on a non-tardis triangulation as a spin network on the boundary by a 'reverse Moussouris algorithm.' We then describe how this procedure is altered for handlebodies of arbitrary genus. From now on, $\Sigma^3$ denotes a handlebody.

**Lemma 1.** *The Ponzano–Regge amplitude for a non-tardis triangulation of $B^3$ can be expressed in terms of a single spin network evaluation*

$$\mathcal{Z}_{\mathrm{PR}}(\Psi, B^3) = \langle (\partial B^3)^* \rangle, \tag{2}$$

*where $^*$ is the two-dimensional dual of the surface triangulation with each dual edge labelled with the $SU(2)$ irrep corresponding to the length of the edge it is dual to, and the spin network is evaluated as the planar projection without crossings, with the intertwiner normalization given by $\Psi$.*

**Proof.** In order to reexpress the 3-ball with a given triangulation $\mathcal{T}$ and the amplitude $\mathcal{Z}(\Psi, B^3)$ as the spin network evaluation of its boundary, we proceed inductively. Note first that a triangulation of $B^3$ given by a single tetrahedron is already of the form we want to put it in: by (1) its amplitude is given exactly by the evaluation of the spin network dual to its boundary 2-geometry with an intertwiner normalization chosen at each surface triangle. This establishes the base case. We now need to show that the statement remains true when one glues tetrahedra on to a ball amplitude already expressed in this manner, and thus reconstruct arbitrary non-tardis triangulations of the 3-ball. To glue we add the necessary face and edge amplitudes for the new interior faces and edges with the same normalization and phase choice of the intertwiners chosen in the boundary state before. These boundary choices will therefore cancel. This is in accordance with the observation above that the phase choice and normalization on the interior are left arbitrary. A tetrahedron can be glued onto a ball non-degenerately with one or two faces.

(1) If we glue one face of the tetrahedron with one face of the ball, we create an inner triangle. The PR amplitude of the new ball differs from the old one by a $\frac{1}{\langle \mathrm{Theta} \rangle}$ and a tetrahedral net. In the spin network evaluation the vertices of the 3-ball and the tetrahedral amplitude corresponding to the glued face, together with the face amplitude, are the normalized

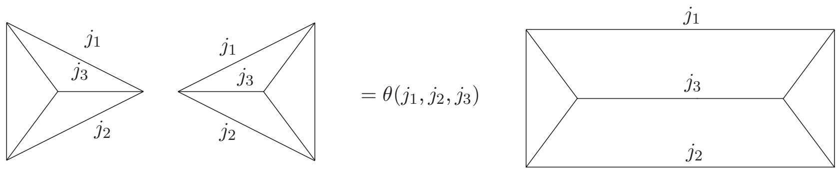


**Figure 1.** Case 1: reduction of the PR amplitude for two tetrahedra to the spin network on the boundary.

projector on the invariant subspace of the irreps on the edges. As both amplitudes being glued already are invariant we can simply replace them with parallel strands, see figure 1. This changes the spin network graph being evaluated by changing a vertex to a triangle. This is the dual to the change of the surface triangulation, and the resulting amplitude still satisfies the lemma.

(2) If we glue two faces of the tetrahedron onto the ball, we create an inner edge and two inner faces, the PR amplitude changes by adding a tetrahedral net, two thetas and one dimension factor. However, these nets correspond exactly to the $6j$ symbol for changing the ball amplitude from being connected along the dual of the old boundary to the dual of the new boundary.

Note that for any non-tardis triangulation of $B^3$, we can always build it up from a single tetrahedron by gluing on one or two faces. Furthermore the two operations described above do not introduce crossings and respect the planar projection chosen.

This establishes that one can express the Ponzano–Regge amplitude of an arbitrary triangulation of $B^3$ as a spin network evaluation on its boundary $S^2$. This proves the lemma. □

Consider next the case of a solid torus $D^2 \times S^1$, which we call $\mathbb{T}$. Take a disc $\mathbb{D} \subset \mathbb{T}$ such that $\mathbb{D} = D^2 \times \{p\} \in D^2 \times S^1$. For future purposes, note that it intersects a non-contractible loop in $\mathbb{T}$. We can now always move this disc by a homotopy that keeps $\partial\mathbb{D}$ on $\partial\mathbb{T}$ such that its boundary is the union of at least three edges of the boundary triangulation. Due to triangulation invariance we can then choose a triangulation such that $\mathbb{D}$ has no internal vertex. We can then cut the Ponzano–Regge amplitude along this surface, the resulting space is topologically $B^3$ and we can apply the previous lemma. This yields a ball where two discs on the boundary are glued by identifying edges and using the PR face and edge weights. Call $n$ the number of edges that make up $\partial\mathbb{D}$. As we chose a disc with no internal vertex, the spin network dual to it has to be an $n-2$ vertex string with one outgoing edge per inner vertex, and two at the end. Together with the face amplitudes this is simply the projector onto the invariant subspace of the irreps on the circle $\partial\mathbb{D}$. This projection can then be replaced by a group averaging on the strands dual to $\partial\mathbb{D}$:

$$\mathcal{Z}_{\mathrm{PR}}(\Psi, \mathbb{T}) = \int_{SU(2)} \mathrm{d}h \langle (\partial\mathbb{T})^*_h \rangle, \tag{3}$$

where $(\partial\mathbb{T})^*_h$ is the spin network dual to the surface of the torus with $h$ inserted along dual edges crossing $\partial\mathbb{D}$. The diagram is defined by first cutting along this circle, choosing the planar no crossing diagram of the graph and then connecting up along the identified edges. See figure 2, and the appendix for an explicit example.

We can easily generalize this example to arbitrary genus handlebodies. By definition, a handlebody of genus $g$ comes equipped with a set of $g$ standard cuts that reduce the handlebody

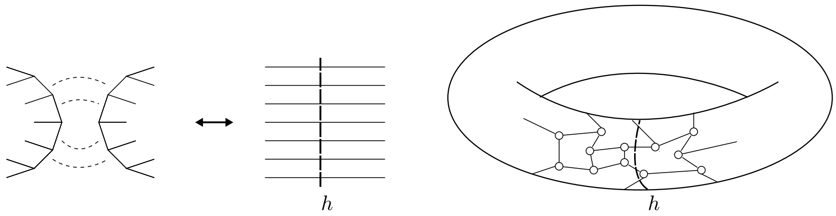


**Figure 2.** By replacing the gluing along the four faces with a group averaging on edges crossing the dashed line (left), we re-express the $\mathbb{T}$ amplitude on the boundary (right).

to the 3-ball. We call these cuts $\mathbb{D}_i$ with an index $i \in C$, where $C$ is a set of labels for the standard cuts. For later use, note that it is always possible to define a complete set of generators $c_i$ of the homology group $H_1(\Sigma^3)$ such that each $c_i$ is transversal to the cut $\mathbb{D}_i$ and does not intersect the other cuts.

We can choose an equivalent set of cuts that are related to the standard cuts by boundary preserving homotopy as long as the cuts remain non-intersecting. In particular from now on we will choose the cuts so as to lie on the triangulation. This implies a restriction on the class of triangulations considered as such a choice may not exist for small triangulations.

Now we can state

**Lemma 2.**

$$\mathcal{Z}_{\mathrm{PR}}(\Psi, \Sigma^3) = \int_{SU(2)} \prod_{i\in C} \mathrm{d}h_i \langle (\partial\Sigma^3)^*_{h_i}\rangle, \tag{4}$$

*where $\Sigma^3$ is a handlebody, $\partial\Sigma^3$ is its triangulated boundary which carries half-integer labels on its edges and $C$ labels the cuts. Choose a set of cuts $\mathbb{D}_i$ that lie on the triangulation. $\langle (\partial\Sigma^3)^*_{h_i}\rangle$ is the spin network evaluation of the dual of the triangulation of the surface, with the links labelled by the half-integer lengths of the edges they cross and an $h_i \in SU(2)$ inserted on every link that crosses a cut $\partial\mathbb{D}_i \in \partial\Sigma^3$, $i \in C$. The spin network is evaluated in the planar projection of the boundary of the cut manifold. That is, with all crossings being due to the links crossing a cut.*

**Proof.** Cutting $\Sigma^3$ along the discs $\mathbb{D}_i$ reduces it to a 3-ball. The spin network evaluation is defined by taking the planar no crossing representation of the graph cut along the circles $\partial\mathbb{D}_i$ and then connecting the identified open ends. If we choose a triangulation that triangulates each disc $\mathbb{D}_i$ without internal vertices and reexpress the resulting amplitude as a spin network evaluation, then the gluing of the faces corresponds to a projection onto invariant subspaces. Replace the projection onto the invariant subspace by a group integration and we get the lemma. □

Note that due to the intertwining property of the spin network the choice of $\mathbb{D}_i$ does not matter, it merely moves the $h_i$ insertion in the intertwiner around.

### 2.2. *Coherent triangles*

In order to have a clear geometric picture of the amplitude we will choose the intertwiners in the boundary state $\Psi$ by using coherent states $\alpha_k(\mathbf{n}, \theta)$ [16]. These are the highest weight eigenstates of the normalized Lie algebra elements, that is for $L^i = \frac{\mathrm{i}}{2}\sigma^i$ the Lie algebra generators and $\mathbf{n} \in S^2$, a coherent state $\alpha_k(\mathbf{n}, \theta)$ in the $k$ representation satisfies

$$L \cdot \mathbf{n}\alpha_k(\mathbf{n}, \theta) = \mathrm{i}k\alpha_k(\mathbf{n}, \theta). \tag{5}$$

The parameter $\theta$ describes a choice of representative of the $U(1)$ equivalence class of states that correspond to the same $\mathbf{n}$. These states transform with a phase under the group elements generated by $L \cdot \mathbf{n}$ and the label $\mathbf{n}$ transforms covariantly under the $SO(3)$ action of $SU(2)$. That is, for $g \in SU(2)$ with the corresponding $SO(3)$ element $\hat{g}$:

$$g\alpha_k(\mathbf{n}, \theta) = \mathrm{e}^{\mathrm{i}k\phi}\alpha_k(\hat{g}\mathbf{n}, \theta). \tag{6}$$

For the asymptotic analysis, three further properties will be crucial.

- The $k$ representation can be constructed as the symmetric subspace of $2k$ copies of the fundamental representation. In this picture coherent states decompose into a tensor product of coherent states in the fundamental representation. Consequently the group action factorizes

$$g\alpha_k(\mathbf{n}, \theta) = g \bigotimes_{i=1}^{2k} \alpha_{\frac{1}{2}}(\mathbf{n}, \theta) = \bigotimes_{i=1}^{2k} \mathrm{e}^{\mathrm{i}\frac{\phi}{2}}\alpha_{\frac{1}{2}}(\hat{g}\mathbf{n}, \theta). \tag{7}$$

- The modulus squared of the Hermitian inner product of coherent states is given by

$$|\langle\alpha_k(\mathbf{n}_1, \theta_1), \alpha_k(\mathbf{n}_2, \theta_2)\rangle|^2 = \left(\tfrac{1}{2}(1 + \mathbf{n}_1 \cdot \mathbf{n}_2)\right)^{2k}. \tag{8}$$

- Under the action of the standard antilinear structure on $SU(2)$ (see [9]), the coherent state changes as

$$L \cdot \mathbf{n}J\alpha_k(\mathbf{n}, \theta) = -\mathrm{i}kJ\alpha_k(\mathbf{n}, \theta). \tag{9}$$

  The antilinear map $J$ is given by multiplication by the epsilon tensor in the spin $k$ representation followed by complex conjugation. $J$ commutes with $SU(2)$ elements.

Note that given a set of three edge labels $k_i$ there is a non-zero intertwiner exactly if they satisfy the triangle inequalities. Therefore, there is a set of $\mathbf{n}_i$, unique up to $O(3)$ such that $\sum_{i=1}^{3} k_i\mathbf{n}_i = 0$. We can then choose our intertwiner in the boundary state $\Psi$ as

$$\iota = \int_{SU(2)} \mathrm{d}X \bigotimes_i (X\alpha_k(\mathbf{n}_i, \theta_i)). \tag{10}$$

This state is clearly an $SU(2)$ invariant state. As we noted that $SU(2)$ acts covariantly as $SO(3)$ on the labels $\mathbf{n}_i$ this choice is only dependent on an unspecified phase as we left open which eigenstates of $L.\mathbf{n}_i$ we are using. In particular, it does not depend on the remaining parity $P = O(3)/SO(3)$ as this acts on the plane of the triangle as an $SO(3)$ element.

Thus choosing normalized $\alpha_k(\mathbf{n}, \theta)$ compatible with the boundary spin labels fixes the intertwiner states up to a parity choice and up to a phase. These two data will be fixed by considering the gluing of the boundary.

### 2.3. Regge state

Let $V$ be a set of labels for the boundary faces. Then we label the boundary edges by pairs $ab \mid a, b \in V$ and call the set of such pairs $E$. Let $\phi_a : \Delta_a \to \mathcal{N}^\perp$ be an orientation-preserving map from the $a$th triangle on $\partial\Sigma^3$ to the plane orthogonal to the north pole of $S^2$ (which we denote $\mathcal{N} = (0, 0, 1)$). We choose the orientation in $\mathcal{N}^\perp$ to be the one inherited from $\mathbb{R}^3$ by taking $\mathcal{N}$ to be the outward surface normal. As the boundary of $\Sigma^3$ is orientable, we can define $\mathbf{n}_{ab} = \phi_a(e_{ab})$ where $ab \in E$.

The requirement that $\phi_a$ be orientation preserving implies that the triangles with the edge vectors given by $k_i \mathbf{n}_i$ all have the same orientation in $\mathcal{N}^\perp$. In particular, we can require them to have the same orientation as we have chosen for $\mathcal{N}^\perp$. In particular, this implies that we can glue up any two triangles $a$,$b$ with a common edge in $\mathcal{N}^\perp$ in an orientation-preserving way.

Thus there exists an element $\hat{g}_{ab} \in SO(3)$ such that

$$-\mathbf{n}_{ba} = \hat{g}_{ab}\mathbf{n}_{ab} \qquad \mathcal{N} = \hat{g}_{ab}\mathcal{N}, \tag{11}$$

where $\mathbf{n}_{ab}$ is the edge vector of triangle $a$ that gets glued to triangle $b$. As in [9], $\hat{g}_{ab}$ is the Levi-Civita parallel translation from triangle $a$ to triangle $b$, according to the bases provided by $\phi_a$ and $\phi_b$. Again, given a choice of spin structure for $\Sigma^3$, a choice of a spin frame for each triangle defines the $SU(2)$ lift $g_{ab}$ as the parallel translation of the spin connection in these frames.

Next we will describe a canonical choice of phase for the boundary state $\Psi$. From (6), $\alpha_k(-\mathbf{n}_{ab}, \theta_{ab})$ is proportional to $g_{ab}\alpha_k(\mathbf{n}_{ba}, \theta_{ba})$. Then we fix the relative phase of the coherent states on the boundary:

$$J\alpha_k(\mathbf{n}_{ba}, \theta_{ba}) = g_{ab}\alpha_k(\mathbf{n}_{ab}, \theta_{ab}). \tag{12}$$

We call coherent states with the above relative state choice Regge states, and denote them as $|\mathbf{n}, k\rangle$. Their image under the antilinear structure is $|-\mathbf{n}, k\rangle = J\,|\mathbf{n}, k\rangle$, and states in the fundamental representation are denoted $|\mathbf{n}\rangle$.

The total boundary state is then given by

$$\Psi(k_i, \mathbf{n}_i) = \int \left(\prod_{a\in V} \mathrm{d}X_a\right) \bigotimes_{cd\in E} X_c\ |\mathbf{n}_{cd}, k_{cd}\rangle. \tag{13}$$

Due to the presence of the antilinear map in the definition of the relative phase the overall ambiguity not fixed by (12) cancels in the overall state. At each triangle $a$, we have a sign freedom as adding a sign contributes $(-1)^{2\sum_{b,ab\in E} k_{ab}} = 1$ by the admissibility conditions on intertwiners. This shows that as in [9] the possible lifts of $\hat{g}_{ab}$ are defined by the spin structures on the boundary and do not rely on the arbitrary spin frame covering chosen to define the lift.

Finally note that inverting the orientation of $\mathcal{N}^\perp$ would have the same effect as turning the state into $\Psi' = J\Psi$.

### 2.4. The amplitude

We begin with $B^3$. To evaluate the spin network defining our amplitude in terms of these coherent intertwiners, we choose a particular diagrammatic representation of the planar graph. To obtain the spin network evaluation of this graph, we then contract the intertwiners chosen using the epsilon inner product defined in terms of the Hermitian inner product by $(\alpha, \beta) = \langle J\alpha|\beta\rangle$. Number the triangles in the graph from left to right. We then assume that the coherent intertwiners have been specified with respect to this planar representation of the graph as well. Then we have no crossings in the diagram and we can now explicitly write the contraction of coherent intertwiners as

$$\begin{aligned}\mathcal{Z}_{\mathrm{PR}}(\Psi, B^3) &= \int \prod_{a\in V} \mathrm{d}X_a \prod_{bc\in E} (X_b\,|\mathbf{n}_{bc}, k_{bc}\rangle\,, X_c\,|\mathbf{n}_{cb}, k_{bc}\rangle)\\ &= \int \prod_{a\in V} \mathrm{d}X_a \prod_{bc\in E} \langle -\mathbf{n}_{bc}, k_{bc}|\, X_b^\dagger X_c\,|\mathbf{n}_{cb}, k_{cb}\rangle\\ &= \int \prod_{a\in V} \mathrm{d}X_a \prod_{bc\in E} \langle -\mathbf{n}_{bc}|\, X_b^\dagger X_c\,|\mathbf{n}_{cb}\rangle^{2k_{bc}},\end{aligned} \tag{14}$$

where we have written $|\mathbf{n}_{cb}\rangle$ for $\left|\mathbf{n}_{cb}, \frac{1}{2}\right\rangle$.

For general manifolds we need to make sure that, after we have chosen the circles, the dual edges crossing a circle all have the same orientation relative to the circle. This can be done by using a planar representation that has all the glued discs strictly left or right of each other. Call $\tilde{E}$ the set of edges not crossing circles and $E_j$ the set of edges crossing circle $j \in C$. The amplitude is then given by

$$\mathcal{Z}_{\mathrm{PR}}(\Psi, \Sigma^3) = (-1)^{\chi} \int \prod_{a \in V} \mathrm{d}X_a \prod_{j \in C} \mathrm{d}h_j \prod_{bc \in \tilde{E}} \langle -\mathbf{n}_{bc}| X_b^{\dagger} X_c |\mathbf{n}_{cb}\rangle^{2k_{bc}} \times \prod_{l \in C} \prod_{de \in E_l} \langle -\mathbf{n}_{de}| X_d^{\dagger} h_l X_e |\mathbf{n}_{ed}\rangle^{2k_{de}}, \tag{15}$$

where $(-1)^{\chi}$ is a sign factor incurred in the spin network evaluation when connecting up the glued edges in the spin network evaluation. This can then be written as

$$\mathcal{Z}_{\mathrm{PR}}(\Psi, \Sigma^3) = (-1)^{\chi} \int \prod_{i \in V} \mathrm{d}X_i \prod_{j \in C} \mathrm{d}h_j e^S, \tag{16}$$

with the action given by

$$S = \sum_{ab \in \tilde{E}} 2k_{ab} \ln \langle \mathbf{n}_{ab}| J X_a^{\dagger} X_b |\mathbf{n}_{ba}\rangle + \sum_{l \in C} \sum_{de \in E_l} 2k_{de} \ln \langle \mathbf{n}_{de}| J X_d^{\dagger} h_l X_e |\mathbf{n}_{ed}\rangle. \tag{17}$$

Note that the ambiguity in the logarithm of a complex number does not affect the amplitude.

### 2.5. *Symmetries of the action*

The action (17) has the following symmetries (up to $2\pi \mathrm{i}$).

- *Continuous.* A global rotation $Y \in SU(2)$ acting on each $X_a$ and $h_i$ as $X_a \to Y X_a$ and $h_i \to Y h_i Y^{-1}$. This represents a rigid motion of the whole manifold.
- *Discrete.* At each triangle $a$, the transformation $X_a \to \epsilon_a X_a$ with $\epsilon_a = \pm 1$ leaves a factor $\epsilon_a^{\sum_{b, ab \in E} 2k_{ab}}$. As the admissibility conditions are satisfied on each triangle, this factor equals 1. Similarly we have an arbitrary sign $\epsilon_i$ on $h_i$ as the edges on which $h_i$ act satisfy the admissibility condition for intertwiners.

This latter symmetry will be used to compensate for the ambiguity of the lifts of $SO(3)$ to $SU(2)$.

### 2.6. *Relation to the standard intertwiner phase choice*

The standard choice of phase for an intertwiner, defined by chromatic evaluation [17], gives real numbers for a spin network evaluation. We will now show that with the Regge phase choice the amplitude is real so can only differ from the chromatic evaluation by $\pm 1$ and a normalization factor. Note that since the Regge choice has all the $\mathbf{n}_{ab}$ orthogonal to $\mathbf{e}_z$, the rotation $\mathrm{e}^{-\mathrm{i}\pi \mathbf{e}_z \cdot \sigma}$ rotates $\mathbf{n}_{cb}$ to $-\mathbf{n}_{cb}$ and leaves $\mathbf{e}_z$ invariant. Under this rotation, the coherent state $|\mathbf{n}_{cb}\rangle$ will transform as

$$\mathrm{e}^{-\mathrm{i}\pi \mathbf{e}_z \cdot \sigma} |\mathbf{n}_{cb}\rangle = \mathrm{e}^{\mathrm{i}\phi} J |\mathbf{n}_{cb}\rangle \tag{18}$$

for some phase $\phi$.

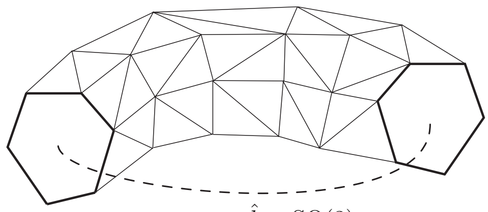


**Figure 3.** A cut immersion for a particular boundary triangulation of a torus. The cutting circles are shown in bold and there is an $\hat{h} \in SO(3)$ that identifies them.

Consider a single term in the amplitude (15), and rewrite it inserting the identity

$$\begin{aligned}\langle -\mathbf{n}_{bc}|\, X_b^{\dagger} X_c \,|\mathbf{n}_{cb}\rangle &= \langle -\mathbf{n}_{bc}|\, \mathrm{e}^{\mathrm{i}\pi \mathbf{e}_z\cdot\sigma} \big(\mathrm{e}^{-\mathrm{i}\pi \mathbf{e}_z\cdot\sigma} X_b^{\dagger}\big)\big(X_c\, \mathrm{e}^{\mathrm{i}\pi \mathbf{e}_z\cdot\sigma}\big)\, \mathrm{e}^{-\mathrm{i}\pi \mathbf{e}_z\cdot\sigma} \,|\mathbf{n}_{cb}\rangle \\ &= \langle \mathbf{n}_{bc}|\, J^{\dagger}\, \mathrm{e}^{\mathrm{i}\pi \mathbf{e}_z\cdot\sigma} \tilde{X}_b^{\dagger} \tilde{X}_c\, \mathrm{e}^{-\mathrm{i}\pi \mathbf{e}_z\cdot\sigma} g_{cb} J \,|\mathbf{n}_{bc}\rangle \\ &= \langle \mathbf{n}_{bc}|\, J^{\dagger}\, \mathrm{e}^{-\mathrm{i}\phi} J^{\dagger} \tilde{X}_b^{\dagger} \tilde{X}_c g_{cb} J\, \mathrm{e}^{\mathrm{i}\phi} J \,|\mathbf{n}_{bc}\rangle \\ &= \langle \mathbf{n}_{bc}|\, J^{\dagger} J^{\dagger} \tilde{X}_b^{\dagger} \tilde{X}_c J \,|\mathbf{n}_{cb}\rangle \\ &= \overline{\langle -\mathbf{n}_{bc}|\, \tilde{X}_b^{\dagger} \tilde{X}_c \,|\mathbf{n}_{cb}\rangle}, \end{aligned} \tag{19}$$

where we have defined the transformation $\tilde{X}_c = X_c\, \mathrm{e}^{\mathrm{i}\pi \mathbf{e}_z\cdot\sigma}$, which can be absorbed on the group integration in (15) and the fact that $J\,|\mathbf{n}_{cb}\rangle = |-\mathbf{n}_{cb}\rangle$. We have used the Regge phase choice (12) from going from the first to the second line and the fact that the $SU(2)$ transformations are all in fact in the same $U(1)$ subgroup (and hence commute). From going to the second to the third line, we have noted that we are acting with opposite rotations on the same state. Hence we get that $\mathcal{Z}_{\mathrm{PR}}(\Psi, \Sigma^3) = \overline{\mathcal{Z}_{\mathrm{PR}}(\Psi, \Sigma^3)}$ which is thus real.

## 3. Asymptotic formula

We wish to study the semiclassical limit of the amplitude $Z_{\mathrm{PR}}(\Psi, \Sigma^3)$. In order to do this, we homogeneously rescale the spin labels by a factor $\lambda$. The corresponding boundary state $\Psi_\lambda$ is given by $\Psi_\lambda = \Psi(\lambda k_i, \mathbf{n_i})$.

Given a set $\mathcal{B} = \{\mathbf{n}_{ab}, k_{ab}\}_{a\neq b}$ of boundary data we denote as $\mathfrak{I}$ the set of cut immersions of the polyhedral surface $\partial\Sigma^3$ with edge lengths $k_{ab}$ in $\mathbb{R}^3$ up to rigid motion.

A *cut immersion* $\mathrm{i} \in \mathfrak{I}$ is an immersion of the manifold obtained from $\partial\Sigma^3$ by the trivializing cuts $\partial\mathbb{D}_i$, $i \in C$, i.e. it is an immersion $\imath(\partial\Sigma^3 - \{\cup_{i\in C}\partial\mathbb{D}_i\}) \hookrightarrow \mathbb{R}^3$. Furthermore, we require the existence of $SO(3)$ elements that identify the two sides of the cut, i.e. $\hat{h}_i \in SO(3)$ such that $\hat{h}_i(\imath(\partial\mathbb{D}_i^+)) = \imath(\partial\mathbb{D}_i^-)$, where $\partial\mathbb{D}_i^-$ and $\partial\mathbb{D}_i^+$ are the elements of the boundary $\partial(\partial\Sigma^3 - \partial\mathbb{D}_i)$ created by the removal of $\partial\mathbb{D}_i$ from $\partial\Sigma^3$ [1].

Any two cut immersions of $\Sigma^3$ are defined to be equivalent and can be obtained from each other if the cuts are related by a homotopy on the surface. Therefore, different choices of cuts $\mathbb{D}_i$ lead to equivalent cut immersions. An example of a cut immersion is given in figure 3.

Such an immersion is called rigid if every continuous deformation of it requires changing the edge lengths, and flexible otherwise. We denote the subset of rigid immersions $\mathfrak{I}_r \subset \mathfrak{I}$.

[1] Note that since $\mathbb{D}_i$ is transversal to generators of $H_1(\Sigma^3)$, its removal changes the connectivity of $\Sigma^3$ and creates two new boundaries $\mathbb{D}_i^-$ and $\mathbb{D}_i^+$.

Through every immersion in $\mathfrak{I}$ passes at least one manifold (with dimension $d$) of immersions that can be continuously deformed into each other. We call these *flexifolds* and denote them $\mathfrak{f}$, we denote the set of flexifolds $\mathfrak{F}$. We then define $\mathfrak{F}_{\max}$ to be the set of flexifolds in $\mathfrak{F}$ of maximal dimension $d_{\max}$. With this definition the rigid immersions are a special case of a flexifold with dimension $d = 0$. We assume from now on that the flexifolds $\mathfrak{f}$ do not intersect.

In the limit $\lambda \to \infty$ we have the following theorem:

**Theorem 1** (Asymptotic formula)**.**

*(1) If* $\mathfrak{I}$ *is not empty we have that*

$$\mathcal{Z}_{\mathrm{PR}}(\psi_\lambda, \Sigma^3) = \left(\frac{2\pi}{\lambda}\right)^{\frac{3(|V|+|C|-1)-d_{\max}}{2}} \sum_{\mathfrak{f}\in\mathfrak{F}_{\max}} N_{\mathfrak{f}} \cos\left(\lambda \sum_{ab\in E} k_{ab}\Theta^{\mathfrak{f}}_{ab} + \phi^{\mathfrak{f}}_{ab}\right) + O\left(\left(\frac{1}{\lambda}\right)^{\frac{3(|V|+|C|)-d_{\max}}{2}}\right). \tag{20}$$

*The coefficient* $N_{\mathfrak{f}}$, *the dihedral angle* $\Theta^{\mathfrak{f}}_{ab}$ *and the phase* $\phi^{\mathfrak{f}}_{ab}$ *are independent of* $\lambda$. *The* $\phi^{\mathfrak{f}}_{ab}$ *and the dihedral angle* $\Theta^{\mathfrak{f}}_{ab}$ *are evaluated on an arbitrary immersion* $\mathfrak{i}$ *in* $\mathfrak{f}$. *It can be shown that these are independent of the cuts. Thus for any particular edge we can evaluate the dihedral angle by moving the cut away from it.* $|V|$ *is the number of triangles (or equivalently vertices in the set V) and* $|C|$ *is the number of cutting circles.* $d_{\max}$ *is the dimension of the flexifolds* $\mathfrak{f} \in \mathfrak{F}_{\max}$, *and* $N_{\mathfrak{f}}$ *now also contains an integral over the union of flexifolds in* $\mathfrak{f}$.

*(2) If no immersions in* $\mathbb{R}^3$ *exist the amplitude is exponentially suppressed:*

$$\mathcal{Z}_{\mathrm{PR}}(\psi_\lambda, \Sigma^3) = o(\lambda^{-n}) \quad \forall n. \tag{21}$$

Note that in the simple case where the boundary data only admits rigid immersions, i.e. if $d_{\max} = 0$, then the sum becomes a sum over the rigid immersions $\mathfrak{i} \in \mathfrak{I}_r$ and we have that

$$\mathcal{Z}_{\mathrm{PR}}(\psi_\lambda, \Sigma^3) = \left(\frac{2\pi}{\lambda}\right)^{\frac{3(|V|+|C|-1)}{2}} \sum_{\mathfrak{i}\in\mathfrak{I}_r} N_{\mathfrak{i}} \cos\left(\lambda \sum_{ab\in E} k_{ab}\Theta^{\mathfrak{i}}_{ab} + \phi^{\mathfrak{i}}_{ab}\right) + O\left(\left(\frac{1}{\lambda}\right)^{\frac{3(|V|+|C|-1)}{2}+1}\right) \tag{22}$$

since $d_{\max} = 0$. Since the immersions are now rigid, the coefficient $N_{\mathfrak{i}}$, the dihedral angles $\Theta^{\mathfrak{i}}_{ab}$ and the phase $\phi^{\mathfrak{i}}_{ab}$ are evaluated on the cut immersion $\mathfrak{i}$.

## 4. Proof of the asymptotic formula

We now prove the above theorem. We begin by describing the methods used to give the asymptotic form of the amplitude; this will require finding the so-called stationary and critical points of the action. We can then interpret these points geometrically and give the asymptotic formula. Much of this section is similar to [9] but one dimension lower so the analysis will be as brief as possible.

### *4.1. Stationary phase*

To find the asymptotic form of the amplitude, we will use the complex stationary phase formula [18], as in [9]. However, we will also require the stationary phase formula for non-isolated critical points. We briefly describe this alteration of the standard stationary phase formula below.

Let $D$ be a closed manifold of real dimension $n$, and let $S$ and $a$ be smooth, complex, valued functions on $D$ such that the real part $\mathrm{Re}\, S \leqslant 0$. Note that $S$ need not be holomorphic. Consider the function

$$f(\lambda) = \int_D \mathrm{d}x\, a(x)\, \mathrm{e}^{\lambda S(x)}. \tag{23}$$

If the action has degenerate critical points, i.e. the Hessian matrix has zero determinant, care is needed to compute the asymptotics (see e.g. [19]). Let $\mathcal{C} := \{y \in D | \delta S(y) = 0, \mathrm{Re} S(y) = 0\}$ denote the set of critical points. Note that we cannot *a priori* assume $\mathcal{C}$ to be a disjoint union of manifolds; however, here we have restricted ourselves to this case so that the following generalized stationary phase theorem applies.

For a smooth function $S$ whose critical set $\mathcal{C}$ is a disjoint union of closed manifolds[2], each critical manifold $\mathcal{C}_{x_0}$ of dimension $p$, labelled by some $x_0$ on the critical manifold, contributes the following to the asymptotic formula [19]:

$$\left(\frac{2\pi}{\lambda}\right)^{(n-p)/2} \mathrm{e}^{\lambda S(x_0)} \int_{\mathcal{C}_{x_0}} \mathrm{d}\sigma_{\mathcal{C}_{x_0}}(y) \frac{a(y)}{\sqrt{\det(-H^\top(y))}} [1 + O(1/\lambda)], \tag{24}$$

where $H^\top(y)$ is the restriction of the matrix to the directions normal to $\mathcal{C}_{x_0}$ with respect to some Riemannian metric on the domain, and $\mathrm{d}\sigma_{\mathcal{C}_{x_0}}$ is the canonical measure induced on the critical submanifold by the same Riemannian measure on the domain space. This extends to the case where $\mathcal{C}$ is a manifold-with-boundary.

*4.1.1. Critical points.* As described above, we must find the points of the action (17) such that $\mathrm{Re} S = 0$ as these are the only points that contribute in the limit $\lambda \to \infty$. First, we introduce some more notation. The action of the elements $X_b$ on the coherent states will produce a new coherent state

$$|\mathbf{n}'_{ab}\rangle = X_a |\mathbf{n}_{ab}\rangle. \tag{25}$$

We will denote the corresponding rotated 3-vectors by

$$\mathbf{n}'_{ab} = \hat{X}_a \mathbf{n}_{ab}, \tag{26}$$

where $\hat{X}_a$ is the $SO(3)$ element corresponding to $X_a$.

We will first consider critical points for edges that are not on one of the cutting circles. Using (8), we can see that the real part of the action is given by

$$\mathrm{Re}\, S = \sum_{ab \in \tilde{E}} k_{ab} \ln \frac{1}{2}(1 - \mathbf{n}'_{ab} \cdot \mathbf{n}'_{ba}). \tag{27}$$

This does not depend on the coherent state phases as it is real. Using this formula, we can see that $\mathrm{Re} S = 0$ when $\mathbf{n}'_{ab} = -\mathbf{n}'_{ba}$ for all $ab$, or explicitly in terms of $SO(3)$ rotations

$$\hat{X}_a \mathbf{n}_{ab} = -\hat{X}_b \mathbf{n}_{ba}. \tag{28}$$

The critical points for an edge that crosses a cutting circle $i$ differ by the inclusion of the $h_i$:

$$\hat{X}_a \mathbf{n}_{ab} = -\hat{h}_i \hat{X}_b \mathbf{n}_{ba}. \tag{29}$$

[2] In the literature this is called a Morse–Bott function. A Morse function is the special case where the critical manifolds are zero-dimensional (so the Hessian at critical points is non-degenerate in every direction, i.e. has no kernel).

*4.1.2. Stationary points.* The stationary points are found by varying the action with respect to each of the group variables $X_a$. The variation of an $SU(2)$ group variable and its inverse is

$$\delta X = TX \qquad \delta X^{-1} = -X^{-1}T \tag{30}$$

for an arbitrary $\mathfrak{su}(2)$ Lie algebra element $T = \frac{1}{2}\mathrm{i}T^j\sigma_j$. The stationary points are given by $\delta S = 0$ and lead to the following equation:

$$\sum_{b:b\neq a} k_{ab}\mathbf{V}_{ab} = 0, \tag{31}$$

where

$$\mathbf{V}_{ab} = \frac{\langle -\mathbf{n}_{ab}|X_a^{-1}\sigma X_b|\mathbf{n}_{ba}\rangle}{\langle -\mathbf{n}_{ab}|X_a^{-1}X_b|\mathbf{n}_{ba}\rangle}. \tag{32}$$

These equations can then be evaluated at the critical points to give

$$\sum_{b:b\neq a} k_{ab}\mathbf{n}_{ab} = 0, \tag{33}$$

which is the closure constraint for an immersed triangle.

The stationary phase condition for the $h_i$ variables is the same but in this case we obtain

$$\sum_{ab\in C_i} k_{ab}\mathbf{n}'_{ab} = 0, \tag{34}$$

which is the closure condition for edges on the circle $i$ immersed in $\mathbb{R}^3$. Note that unlike the closure condition for the triangle, this relation involves the $\mathbf{n}'_{ab}$ as each edge belongs to a different triangle.

If the critical points are not isolated but form a manifold of critical points, then we denote this manifold by

$$\mathcal{C}_X = \{(X_1, \ldots, X_{|V|}, h_1, \ldots, h_{|C|}) \in SU(2)^{|V|+|C|} : \delta S = 0, \mathrm{Re}(S) = 0\}. \tag{35}$$

### *4.2. Geometric analysis*

We will now describe how the critical/stationary points and the action evaluated at them can be given a geometric interpretation. Each critical point will correspond to an immersion of the boundary data in $\mathbb{R}^3$ and we will show that the action evaluated on the critical points gives the Regge action for that immersion. For triangulations of the 3-ball and integer spin this is straightforward; however, the additional group elements arising from the trivializing cuts of higher genus handlebodies and the need to lift the geometric $SO(3)$ quantities to $SU(2)$ will introduce complications.

We first establish the geometric interpretation of the critical/stationary point equations in terms of immersions of the boundary in $\mathbb{R}^3$. We will then discuss in some details the Regge action for handlebodies in terms of the immersion of their boundary, paying particular attention to the sign of the dihedral angles and the role of the trivializing cuts. We then show that the group elements $X_a$ at the critical points have an interpretation as $SU(2)$ gauge transformations relating the gluing maps and the dihedral rotations interpreted as connections. The discrete symmetries are interpreted as giving spin frames and spin structures. Using this we can precisely evaluate the action at the critical point to the Regge action.

*4.2.1. Geometric correspondence of critical points.* The critical points correspond to immersions of the surface geometry defined by the boundary data cut along the trivializing cuts. This is easy to see as the critical and stationary equations simply enforce the existence of a consistent set of edge vectors to be associated with each edge. Precisely formulated we have that

**Lemma 3** (Geometry)**.** *Given a set of boundary data $\mathcal{B}$ satisfying the closure constraint on each triangle, the solutions $X_a$, $h_i$ to the critical and stationary point equations (28), (29) and (34) correspond to immersions of a geometric triangulated 2-manifold with boundary in $\mathbb{R}^3$. This manifold is the one obtained by cutting the boundary manifold $\partial\Sigma^3$ along the trivializing cuts $C$ and has a boundary $\bigcup_{i\in C}\partial\mathbb{D}_i^+\cup\partial\mathbb{D}_i^-$. This immersion is subject to the constraint that a set of $\hat{h}_i\in SO(3)$ exists that map the immersion of $\partial\mathbb{D}_i^+$ to the parity flipped $P(\partial\mathbb{D}_i^-)$, that is, the immersion of $\partial\mathbb{D}_i^+$ is congruent and oppositely oriented to the immersion of $\partial\mathbb{D}_i^-$ (i.e. figure 3).*

*The edge vectors of the immersion are given by*

$$\mathbf{v}_{ab}(\mathfrak{i})=k_{ab}\hat{X}_a\mathbf{n}_{ab}.$$

*Its orientation is the one induced by the vectors on each face.*

*Conversely, an immersed surface $\mathfrak{i}$ determines a set of $k_{ab}$, $\mathbf{n}_{ab}$ and a set of $SO(3)$ elements $\hat{X}_a(\mathfrak{i})$ up to $SO(2)$ rotations.*

**Proof.** Start somewhere on the surface of $\Sigma^3$ that is not in a circle. Since $\Sigma^3$ has a connected boundary, the entire surface is now contractible to this point if cut along the circles. Take the triangle $\Delta_a$ you are on as embedded in $\mathcal{N}^\perp$ and rotate it according to the stationary point equations to $X_a\Delta_a$. Embed the next triangle and rotate it, according to the stationary point equations its edges are now antiparallel to already immersed edges. As these are geometrically glued a translation exists that identifies all its edges with already immersed ones. Thus iteratively the whole immersed surface can be built up and the closure conditions on the cuts now imply that the surface closes up. Finally the stationarity equations on the circles imply that the $P\hat{h}_i$ identifies the circles where we cut the surface.

Conversely given an oriented immersion with the right edge lengths we can choose a set of edge vectors compatible with the orientation on the surface. On each triangle there are two linearly independent edge vectors. The map from these to the corresponding boundary elements defines a rotation in $SO(3)$. On the boundary circles, we explicitly have $SO(3)$ elements. The complete set of these solves the critical point equations. □

If there is a manifold of dimension $d>0$ of critical points then lemma 3 holds for each critical point in $\mathcal{C}_X$. Since these critical points lie on a manifold, there is a continuous deformation of the immersed surface that does not change the edge lengths. Hence these critical points reconstruct flexible immersions and we arrive at the flexifolds $\mathfrak{f}$ described in section 3. We will now label the critical manifolds by $\mathcal{C}_\mathfrak{f}$, where $\mathfrak{f}$ is the flexifold that it describes.

*4.2.2. The Regge action for handlebodies.* For a convex polyhedral body embedded in $\mathbb{R}^3$ it is straightforward to write down its Regge action. It is simply given by summing over the product of dihedral angles and edge lengths $S_R=\sum l_{ab}\Theta_{ab}$ on the boundary. Here the dihedral angle $\Theta_{ab}$ is defined by $\cos\Theta_{ab}=\mathcal{N}_a\cdot\mathcal{N}_b$ and $0\leqslant\Theta_a<\pi$, where $\mathcal{N}_a$ is the outward facing normal of the triangle $a$. To define the Regge action for the kind of immersions we have defined above we need to be more careful. We can no longer assume $0\leqslant\Theta_a<\pi$ to fix the sign of the dihedral angle, and there is no longer an obvious notion of outward facing normal.

Furthermore we need to take account of the presence of the trivializing cuts. We will here give a description of the Regge action in terms of the cut immersion of its boundary, taking account of these facts. These particular definitions will be well suited for the analysis of the action at the critical points.

Given an oriented surface in $\mathbb{R}^3$, the standard orientation automatically gives us a consistent set of normals $\mathcal{N}_a$ to the $a$th triangles. This replaces the notion of outward facing normals for a convex body.

By our choice of boundary data we have automatically ensured that these normals are given simply by

$$\mathcal{N}_a = \hat{X}_a \mathcal{N}.$$

We can define the dihedral rotation for an oriented surface unambiguously as the rotation $\hat{D}_{ab} \in SO(3)$ around the geometric edge vector $\mathbf{v}_{ab}(\mathrm{i})$ that takes $\mathcal{N}_a$ to $\mathcal{N}_b$. That is

$$\mathcal{N}_b = \hat{D}_{ab} \mathcal{N}_a$$

and

$$\mathbf{v}_{ab}(\mathrm{i}) = \hat{D}_{ab} \mathbf{v}_{ab}(\mathrm{i}).$$

A lift of this rotation to an $SU(2)$ element can thus be written as

$$\begin{aligned} D_{ab} &= \exp\left(\Theta^{\mathrm{i}}_{ab} \frac{\mathbf{v}_{ab}(\mathrm{i})}{|\mathbf{v}_{ab}(\mathrm{i})|}.L\right) \\ &= \exp(\Theta^{\mathrm{i}}_{ab} \mathbf{n}'_{ab}.L) \in SU(2), \end{aligned} \tag{36}$$

where $L$ are $SU(2)$ generators and we require $-\pi < \Theta^{\mathrm{i}}_{ab} \leqslant \pi$. We then call $\Theta^{\mathrm{i}}_{ab}$ the dihedral angle. As $\mathbf{v}_{ab}(\mathrm{i}) = -\mathbf{v}_{ba}(\mathrm{i})$ this definition clearly implies $\Theta^{\mathrm{i}}_{ab} = \Theta^{\mathrm{i}}_{ba}$. It can be checked that this is a lift in the same sense as for the $g_{ab}$, corresponding to a spin structure to be studied later and a choice of spin frame in each triangle. If we have a surface defining a convex subspace of $\mathbb{R}^3$, this definition reduces to the usual definition up to a global sign. In particular, the consistent choice of orientation then ensures that we have $0 \leqslant \Theta^{\mathrm{i}}_{ab} < \pi$ and $\cos\left(\Theta^{\mathrm{i}}_{ab}\right) = \mathcal{N}_a \cdot \mathcal{N}_b$ if the thus defined normals are outward facing and $0 \leqslant -\Theta^{\mathrm{i}}_{ab} < \pi$ if inward.

If we are considering handlebodies with $g > 0$, then the immersed surface will have a boundary. On the boundaries of the surface we can also define an analogue of the dihedral rotation by requiring

$$\hat{h}_i \mathcal{N}_b = \hat{D}_{ab} \mathcal{N}_a$$

and

$$-\hat{h}_i \mathbf{v}_{ba}(\mathrm{i}) = \mathbf{v}_{ab}(\mathrm{i}) = \hat{D}_{ab} \mathbf{v}_{ab}(\mathrm{i}).$$

Geometrically this makes sense as it corresponds to the angle obtained by gluing on the two identified boundaries of the immersed surface. This then allows us to give the Regge action of a cut immersion as

$$S_R = \sum k_{ab} \Theta_{ab}.$$

Given a particular triangulation of the interior of the handlebody without internal vertices, we can define a metric throughout the handlebody by inserting the internal edges into the immersion and pushing forward the metric in the immersed tetrahedra to the handlebody. After all we are defining the metric on the handlebody with the metrics of the tetrahedra right?[3] The Regge action defined here is just the Einstein–Hilbert action evaluated on that

[3] Note that this metric will in general not be a flat metric, and it is possible that no interior triangulation corresponding to a flat metric exists.

metric simplexwise and summed with relative signs given by the pull back of the orientation. Note that though the metric will depend on the particular triangulation chosen, the action defined in this way does not. We can verify that it does not depend on the trivializing cut by direct calculation.

In particular moving a triangle $a$ across the cut has, according to the geometric correspondence established above, exactly the effect of changing $\mathcal{N}_a$ to $h_i^{-1}\mathcal{N}_a$ and $v_{ab}$ to $h_i^{-1}v_{ab}$. Thus, we have $\mathcal{N}_b = \tilde{D}_{ab}h_i^{-1}\mathcal{N}_a$ and $v_{ab} = \tilde{D}_{ab}h_i^{-1}v_{ab}$. Therefore, by direct comparison we have $\tilde{D}_{ab} = h_i^{-1}D_{ab}h_i$ and the dihedral rotation changes only by conjugation, its eigenvalues are unchanged. This shows that the Regge action defined here is indeed invariant under moving the cuts.

*4.2.3. The group variables at the critical points.* The dihedral rotations are given as $SO(3)$ elements; the group elements in the action, however, are $SU(2)$. As before with the gluing maps $g_{ab}$ we have, through a choice of spin structure and spin frames at each triangle, lifted the $\hat{D}_{ab}$ to a set of $D_{ab}$. These again have an interpretation as an $SU(2)$ connection on the boundary. In this section, we will show that the $X_a$ can be interpreted as being a gauge transformation relating the connection $D_{ab}$ to the connection obtained from $g_{ab}$ by inserting $h_i$ whenever crossing a cut. The discrete symmetries are found to account for arbitrary choices of the spin structure and spin frames. This allows us to resolve all sign ambiguities in relating the action of the state sum amplitude to the Regge action. We will proceed by first noting the equivalence as $SO(3)$ connections and then discussing in some detail the lift to $SU(2)$

*Connections.* Consider the following diagram which applies to two adjacent triangles that are not on a cutting circle:

$$\begin{array}{ccc} t_a & \xrightarrow{\;X_a\;} & \tau_a \\ {\scriptstyle g_{ab}}\downarrow & & \downarrow{\scriptstyle (-1)^{\nu_{ab}}D_{ab}} \\ t_b & \xrightarrow[\;X_b\;]{} & \tau_b \end{array} \tag{37}$$

Here $t_a$ is the boundary triangle at $\mathcal{N}^{\perp}$ with edge vectors given by $k_{ab}\mathbf{n}_{ab}$ and $\tau_a$ is the triangle rotated according to its location in the surface, which according to the geometry lemma 3 has edge vectors given by $\mathbf{v}_{ab}(\text{i}) = k_{ab}\hat{X}_a\mathbf{n}_{ab}$. The $SO(3)$ action of this diagram immediately commutes, as can be seen by acting on $\mathbf{n}_{ab}$ and $\mathcal{N}$. This implies that the $X_a$ act as gauge transformations relating the $SO(3)$ connections $\hat{D}_{ab}$ and $\hat{g}_{ab}$ away from the circles. To analyse the lift to $SU(2)$ connections we define a sign $(-1)^{\nu_{ab}}$ that makes it commute as $SU(2)$. The discrete sign symmetry $X_a \to \epsilon_a X_a$ of the action can be seen as acting on this sign by $(-1)^{\nu_{ab}} \to \epsilon_a\epsilon_b(-1)^{\nu_{ab}}$.

Now, for two triangles whose common edge is on a cutting circle $i$, in the same way we have a commuting diagram as

$$\begin{array}{ccc} t_a & \xrightarrow{\;X_a\;} & \tau_a \\ {\scriptstyle g_{ab}}\downarrow & & \downarrow{\scriptstyle (-1)^{\nu_{ab}}D_{ab}} \\ t_b & \xrightarrow[\;h_iX_b\;]{} & \tau_b \end{array} \tag{38}$$

and additionally have $(-1)^{\nu_{ab}} \to \epsilon_a\epsilon_b\epsilon_i(-1)^{\nu_{ab}}$. Together these diagrams can be interpreted as saying that $\hat{D}_{ab}$ is indeed a gauge transformation obtained from the connection given by $\hat{g}_{ab}$ away from the cuts and $\hat{g}_{ab}\hat{h}_i$ on the cut.

*Spin lift.* Now we will fix the signs emerging from the spin lifts of the dihedral angle by exploring the discrete sign symmetry in $h_i$ and $X_a$. Recall that the discrete sign freedom of the action $X_a \to \epsilon_a X_a$ emerged from a different choice of spin frame for each triangle. Now, we show that the discrete sign symmetry related to the cuts $h_i \to \epsilon_i h_i$ corresponds to different choices of spin structures for the manifold $\Sigma^3$. Then, using the fact that $(-1)^{\nu ab} D_{ab}$ is a gauge transform of the connection $g_{ab}$ we can fix the symmetries by adjusting the spin frames and the spin structure such that $(-1)^{\nu_{ab}} = 1$. Thus we will show that

**Lemma 4.** *The signs $(-1)^{\nu_{ab}}$ arising from the spin lift on each face not on the cut obey $(-1)^{\nu_{ab}} = \kappa_{ab} = \kappa_a \kappa_b$ for some $\kappa_a = \pm 1$. The signs for a face on the cut, i.e. $ab \in i \in C$ obey $(-1)^{\nu_{ab}} = \kappa_{ab} = \kappa_a \kappa_b \kappa_i$ where $\kappa_i$ parametrizes the spin structures of $\Sigma^3$.*

**Proof.** First of all, by (4.2.3) a lift of the dihedral rotations, $\kappa_{ab} D_{ab}$, is just a gauge transformation of the $g_{ab}$. Now recall that $\hat{g}_{ab} \in SO(3)$ are parallel translations on the boundary triangles according to the Levi-Civita connection of the associated metric, with $g_{ab}$ being the parallel translation of the respective spin connection (a lift of $\hat{g}_{ab}$ to $SU(2)$).

But when the geometry around a vertex is continuously deformed to the flat geometry, the $g_{ab}$ holonomy of a trivial cycle around said vertex has to go to the identity rotation, as opposed to a $2\pi$ rotation. This implies that for the holonomy around a vertex through triangles $a, b$ and $c$ (which of course consists of a trivial cycle), we have

$$\kappa_{ca} \kappa_{bc} \kappa_{ab} D_{ca} D_{bc} D_{ab} = \kappa_{ca} \kappa_{bc} \kappa_{ab} = 1,$$

which implies that locally we must have $\kappa_{ab} = \kappa_a \kappa_b$. The problem now is that if there are non-trivial cycles, i.e. $g \neq 0$, we may not be able to extend this globally, i.e. $\kappa_{ab}$ may not be globally pure gauge.

In other words, for trivial cycles, the lift of the holonomy given by the $\kappa_{ab} D_{ab}$ is fixed to be the same as that given by $D_{ab}$. But not so for the holonomy of a non-trivial cycle; there exist inequivalent spin structures on a manifold. These have a one-to-one correspondence with the elements of $H_1(\Sigma^3, \mathbb{Z}_2)$, and so are $2^g$ in number. Hence for a non-trivial cycle, dual to the sequence of triangles $\Delta_{a_0} \cdots \Delta_{a_n} \Delta_{a_0}$ crossing the circle $i \in C$, we have

$$\kappa_{a_n a_0} \kappa_{a_{n-1} a_n} \cdots \kappa_{a_0 a_1} D_{a_n a_0} D_{a_{n-1} a_n} \cdots D_{a_0 a_1} = \kappa_i D_{a_n a_0} D_{a_{n-1} a_n} \cdots D_{a_0 a_1}, \tag{39}$$

where a $\kappa_i$ is introduced whenever there is an implicit choice of spin structure, i.e. it parametrizes the different spin structures associated with the cut.

We reconcile this case with the $g = 0$ one by keeping the form $\kappa_{ab} = \kappa_a \kappa_b$ for all the edges $ab$ that do not lie on a circle, i.e. $ab \notin i$ for any $i \in C$. Then by (39) immediately we must have for $ab \in i$, $\kappa_{ab} = \kappa_i \kappa_a \kappa_b$. Since our chosen basis for $H_1(\Sigma^3, \mathbb{Z}_2)$ generates all cycles, we can see that this form of $\kappa_{ab}$ has all the right properties demanded by our equations and accounts for the different spin structures.

Therefore, taking advantage of the discrete sign symmetry, we can choose the spin structure to be compatible with the one chosen for the lift of $g_{ab}$ and thus we will have $(-1)^{\nu_{ab}} \to \epsilon_i \epsilon_a \epsilon_b (-1)^{\nu_{ab}}$ that makes $(-1)^{\nu_{ab}} = 1$. □

*4.2.4. The action at the critical points.* We can now easily evaluate the action at the critical points. We evaluate at the symmetry-related critical point at which $(-1)^{\nu_{ab}} = 1$. By equation (36) we have that

$$(X_a(\mathrm{i}))^{-1} D_{ab} X_a(\mathrm{i}) = \exp\left(\Theta^{\mathrm{i}}_{ab}\, \mathbf{n}_{ab}.L\right). \tag{40}$$

Acting with $X_a^{-1}$ by the left of the commuting diagram equations, using the notation $X_{ab}(\mathrm{i}) = (X_a(\mathrm{i}))^{-1} X_b(\mathrm{i})$ if not on a circle and $X_{ab}(\mathrm{i}) = (X_a(\mathrm{i}))^{-1} h_i X_b(\mathrm{i})$ if on, we get

$$X_{ab}(\mathrm{i}) g_{ab} = (X_a(\mathrm{i}))^{-1} D_{ab} X_a(\mathrm{i}) = \exp(\Theta_{ab}(\mathbf{n}_{ab}.L), \tag{41}$$

where we have used (40).

It is now straightforward to evaluate the matrix elements in the amplitude. These are of the form $\langle -\mathbf{n}_{ab}| X_{ab} |\mathbf{n}_{ba}\rangle$. Using the gluing condition this becomes $\langle -\mathbf{n}_{ab}| X_{ab} g_{ab} |-\mathbf{n}_{ab}\rangle$. Finally, by (41) this is just $e^{\frac{i}{2}\Theta^{\mathfrak{i}}_{ab}}$. Thus we have overall that

$$\langle -\mathbf{n}_{ab}| X_{ab} |\mathbf{n}_{ba}\rangle = e^{\frac{i}{2}\Theta^{\mathfrak{i}}_{ab}}. \tag{42}$$

Finally, we obtain that the action evaluated at the critical points is the Regge action for the immersed surface $\mathfrak{i}$:

$$S = \sum_{ab\in E} k_{ab}\Theta^{\mathfrak{i}}_{ab}. \tag{43}$$

For the flexible immersions, the action is the same for all points[4] on the critical manifold so we evaluate it on an arbitrary immersion in the flexifold.

*Parity.* Note that by the geometry lemma every immersion with the same boundary data gives a solution to the critical point equations. The boundary geometry is left invariant by the action of $O(3)$. Acting by $SO(3)$ on the immersion changes the solution to one related by the continuous symmetries; however, acting with an element in the component of $O(3)$ not connected to the identity creates a new solution. In particular acting with parity $P : \mathbf{n} \to -\mathbf{n}$ switches the sign of the dihedral angle. This is because the two equations defining $D_{ab}$ are invariant under parity, and the dihedral rotation is unchanged. Thus by the definition of the dihedral angle we have

$$D_{ab} = \exp\left(\Theta^{\mathfrak{i}}_{ab}\left(-\frac{\mathbf{v}_{ab}}{|\mathbf{v}_{ab}|}(P\sigma)\right).L\right)$$

and so $\Theta^{P\mathfrak{i}}_{ab} = -\Theta^{\mathfrak{i}}_{ab}$.

Thus after fixing the continuous symmetry we still will always obtain two solutions with complex conjugate action to each other.

### 4.3. Hessian

The stationary phase formula requires us to calculate the Hessian of the action $S$ to determine the weights with which the stationary points contribute to the action. This will be a $3(|V| + |C|) \times 3(|V| + |C|)$ matrix defined by

$$H = \begin{pmatrix} H_{XX} & H_{Xh} \\ H_{hX} & H_{hh} \end{pmatrix}, \tag{44}$$

where

$$(H_{XX})^{ij}_{cd} = \left(\frac{\partial^2 S}{\partial X^i_c \partial X^j_d}\right), \qquad (H_{hX})^{ij}_{pd} = \left(\frac{\partial^2 S}{\partial h^i_p \partial X^j_d}\right),$$
$$(H_{Xh})^{ij}_{cq} = \left(\frac{\partial^2 S}{\partial X^i_c \partial h^j_q}\right), \qquad (H_{hh})^{ij}_{pq} = \left(\frac{\partial^2 S}{\partial h^i_p \partial h^j_q}\right). \tag{45}$$

The global $SU(2)$ symmetry of the action implies that there is a redundant integration in $I$. This will cause the determinant of the Hessian to be zero unless it is gauge fixed. To solve this, we make the change of variables $X_a \to X_b X_a$ for some $b \in \{1,\ldots,|V|\}$. This has the effect of removing the $X_b$ variables and the integral gives a volume of $SU(2)$ which can be normalized

[4] In the mathematical literature this fact is known as the 'strong bellows conjecture for the case of mean curvature' and was shown in [20]. Several proofs and stronger versions are known, see e.g. [21].

to 1 as it is compact. The remaining Hessian is now a $3(|V| + |C| - 1) \times 3(|V| + |C| - 1)$ matrix. The submatrix $H_{XX}$ at the critical points is given by

$$\left(\frac{\partial^2 S}{\partial X_c^i \partial X_c^j}\right)\Bigg|_{\delta S=0,\mathrm{Re}S=0} = \frac{1}{2}\sum_{b\neq c, bc\in E} k_{cb}\big(\delta^{ij} - n'^i_{cb} n'^j_{cb}\big) \tag{46}$$

for the diagonal terms. The off-diagonal part is

$$\left(\frac{\partial^2 S}{\partial X_c^i \partial X_d^j}\right)\Bigg|_{\substack{\delta S=0\\ \mathrm{Re}\, S=0}} = -\frac{1}{2}\sum_e (\delta_{cs(e)}\delta_{dt(e)} + \delta_{ds(e)}\delta_{ct(e)})$$
$$\times \big(\delta^{ij} - \mathrm{i}\epsilon^{ijk} n^k_{s(e)t(e)} - n'^i_{s(e)t(e)} n'^j_{s(e)t(e)}\big). \tag{47}$$

So one can see that only the off-diagonal elements that represent two neighbouring triangles are non-zero. The $(H_{hh})$ submatrix will be diagonal since each term in the action only contains one $h_p$ term (i.e. each dual edge only crosses one cut)

$$\left(\frac{\partial^2 S}{\partial h_p^i \partial h_p^j}\right)\Bigg|_{\delta S=0,\mathrm{Re}S=0} = \frac{1}{2}\sum_{b\neq c, bc\in C_p} k_{cb}\big(\delta^{ij} - n'^i_{cb} n'^j_{cb}\big). \tag{48}$$

The mixed terms $H_{Xh}$, $H_{Xh}$ will be non-zero only for triangles with an edge on the cut:

$$\left(\frac{\partial^2 S}{\partial X_c^i \partial h_q^j}\right)\Bigg|_{\substack{\delta S=0\\ \mathrm{Re}\, S=0}} = -\frac{1}{2}\sum_{\substack{ab\in E_q\\ c=a,b}} k_{ab}\big(\delta^{ij} - i\epsilon^{ijk} n^k_{ab} - n'^i_{ab} n'^j_{ab}\big). \tag{49}$$

Note that

$$(X_a(\mathrm{i}))^{-1} D_{ab} X_a(\mathrm{i}) = \exp\big(-\Theta^{\mathrm{i}}_{ab}(\mathbf{n}_{ab}(\mathrm{i})).L\big)^{-1} = ((X_a(P\mathrm{i}))^{-1} D_{ab} X_a(P\mathrm{i}))^{-1}, \tag{50}$$

where we used that $D_{ab} = \exp\big(\Theta^{\mathrm{i}}_{ab}\big(-\frac{\mathbf{v}_{ab}(P\mathrm{i})}{|\mathbf{v}_{ab}(P\mathrm{i})|}\big).L\big)$ on the second equality. By (41) we then have $(X_{ab}(\mathrm{i})g_{ab}) = (X_{ab}(P\mathrm{i})g_{ab})^{-1}$, so if we replace the $X_{ab}(\mathrm{i})$ in $\langle -\mathbf{n}_{ab}| X_{ab}(\mathrm{i}) |\mathbf{n}_{ba}\rangle$ with the parity-related one we now obtain the complex conjugate

$$\begin{aligned}\langle -\mathbf{n}_{ab}| X_{ab}(\mathrm{i})g_{ab} |-\mathbf{n}_{ab}\rangle &= \langle -\mathbf{n}_{ab}| (X_{ab}(P\mathrm{i})g_{ab})^{-1} |-\mathbf{n}_{ab}\rangle \\ &= \overline{\langle -\mathbf{n}_{ab}| X_{ab}(P\mathrm{i})g_{ab} |-\mathbf{n}_{ab}\rangle} \\ &= \overline{\langle -\mathbf{n}_{ab}| X_{ab}(P\mathrm{i}) |\mathbf{n}_{ba}\rangle}.\end{aligned} \tag{51}$$

Thus we can see that the action of parity on the Hessian matrix will also result in complex conjugation when evaluated at the critical points.

### *4.4. Proof of the formula*

We can now apply the stationary phase approximation to the amplitude $\mathcal{Z}_{\mathrm{PR}}(\Psi_\lambda, \Sigma^3)$ defined in (15). We begin by fixing the symmetries of the action. This can be achieved by taking an arbitrary vertex and dropping the group integration associated with it. As shown in section 4.1.1 the critical point equations are the equations for the immersion of a polyhedral surface with the geometry specified in the boundary data. If a particular immersion is rigid, no infinitesimal deformation taking it to another such immersion exists and therefore it is an isolated critical point of the amplitude.

For the isolated critical points in $\mathfrak{I}_r$ we can explicitly evaluate the stationary phase approximation. Having fixed one group integration we are left with a $3(|V| + |C| - 1)$-dimensional integration. The overall scaling of these points is thus $\left(\frac{2\pi}{\lambda}\right)^{3(|V|+|C|-1)/2}$. Further we obtain a set of $2^{3(|V|+|C|-1)}$ critical points for each immersion from the spin lift of each $SU(2)$. Finally the derivatives in the Hessian as defined above are taken with respect to a parametrization of $SU(2)$ with volume $(4\pi)^2$, so we need to rescale by this factor. Using equation (42) and lemma 4, the amplitude itself evaluates to the Regge action of the cut immersion:

$$\ln \langle -\mathbf{n}_{ab}| X_{ab} |\mathbf{n}_{ba}\rangle^{2k_{ab}} = \mathrm{i}k_{ab}\Theta^{\mathfrak{i}}_{ab}.$$

Since parity complex conjugates the action and since the Hessian matrix changes to its complex conjugate with parity, we can absorb the phase of the determinant into the exponentials and combine the terms from the immersion $\mathfrak{i}$ and the parity-related immersion $P\mathfrak{i}$ into a cosine. Taking all these factors together we can approximate the contributions of the isolated critical points to the partition function as

$$\begin{aligned}\mathcal{Z}_{\mathrm{PR}}(\Psi_\lambda, \Sigma^3) = 2(-1)^\chi \left(\frac{1}{4\pi\lambda}\right)^{\frac{3(|V|+|C|-1)}{2}} \sum_{\mathfrak{i}\in\mathfrak{I}_r} \frac{1}{\sqrt{|\det H_{\mathfrak{i}}|}} \\ \times \cos\left(\mathrm{i}\lambda \sum_{ab\in E} k_{ab}\Theta^{\mathfrak{i}}_{ab} - \frac{1}{2}\mathrm{Arg}(\det H_{\mathfrak{i}})\right) \\ + O\left(\left(\frac{1}{\lambda}\right)^{\frac{3(|V|+|C|-1)}{2}+1}\right),\end{aligned} \tag{52}$$

where $\Theta^{\mathfrak{i}}_{ab}$ is the dihedral angle of the edge $ab$ in the cut immersion $\mathfrak{i} \in \mathfrak{I}_r$.

If there are any flexible immersions of the boundary data then there will be a manifold of critical points. Since the critical points extremize the action, it must have the same value on every point of the critical manifold. The Hessian therefore has zero modes along the directions of the flexifold and we must treat the integral as having further symmetries in the neighbourhood of the flexifold. Factoring out these changes the scaling of the contribution of these critical points by $\lambda^{d_{\max}/2}$, where $d_{\max}$ is the dimension of the flexifold. Therefore, these immersions dominate the rigid immersions if they exist; their contribution is given by

$$\begin{aligned}\mathcal{Z}_{\mathrm{PR}}(\Psi_\lambda, \Sigma^3) = (-1)^\chi \left(\frac{2\pi}{\lambda}\right)^{\frac{3(|V|+|C|-1)-d_{\max}}{2}} \left(\frac{2}{(4\pi)^2}\right)^{\frac{3(|V|+|C|-1)-d_{\max}}{2}} \\ \times \left[\sum_{\mathfrak{f}\in\mathfrak{F}_{\max}} L^{\mathfrak{f}} \exp\left(\mathrm{i}\lambda \sum_{ab\in E} k_{ab}\Theta^{\mathfrak{f}}_{ab}\right) + \overline{L^{\mathfrak{f}}} \exp\left(-\mathrm{i}\lambda \sum_{ab\in E} k_{ab}\Theta^{\mathfrak{f}}_{ab}\right)\right] \\ + O\left(\left(\frac{1}{\lambda}\right)^{\frac{3(|V|+|C|-1)-d_{\max}}{2}+1}\right),\end{aligned} \tag{53}$$

where $\Theta^{\mathfrak{f}}_{ab}$ is the dihedral angle of the edge $ab$ of a particular cut immersion $\mathfrak{i}$ in the flexifold $\mathfrak{f}$. As the action is constant along the flexifold it does not matter where we evaluate it. $L^{\mathfrak{f}}$ is given by

$$L^{\mathfrak{f}} = \int_{\mathcal{C}_{\mathfrak{f}}} \mathrm{d}\sigma_{\mathcal{C}_{\mathfrak{f}}}(y) \frac{a(y)}{\sqrt{\det H^{\top}_{\mathfrak{f}}(y)}}, \tag{54}$$

where $H^{\top}_{\mathfrak{f}}$ is the Hessian matrix for the transverse directions which we cannot give a general formula for. Combining the exponentials into cosines as above we obtain part 1 of the main theorem.

Finally, if no immersions of the boundary data exist, then there are no solutions to the critical point equations and the stationary phase formula results in a suppressed amplitude.

## 5. Example: the tetrahedron

Here we apply the above results to the well-known case of the asymptotics of the amplitude for a single tetrahedron which, with an appropriate choice of normalization for the boundary intertwiners, will correspond to the $6j$ symbol. This is a special case of theorem 1 so the proof is the same as above. In particular, the critical and stationary point equations are the same and the action evaluated at these points reduces to the Regge action for a tetrahedron. Since the asymptotic formula for the tetrahedron is already known, we must verify that our formula agrees with this result. This also provides further evidence that the asymptotic formula for the 4D case derived using the same methods in [9] is correct. We begin by noting that, up to parity, the boundary data of a tetrahedron has only one immersion so the sum in the asymptotic formula disappears.

### *5.1. Normalization and scaling behaviour*

We can now compare our theorem with the Ponzano–Regge asymptotic formula for the $6j$ symbol. The Ponzano–Regge formula is

$$\begin{Bmatrix} \lambda k_{12} & \lambda k_{13} & \lambda k_{14} \\ \lambda k_{23} & \lambda k_{24} & \lambda k_{34} \end{Bmatrix} \to \frac{1}{\sqrt{12\pi \,\mathrm{Vol}}} \cos\left(\sum_{a<b}\left(\lambda k_{ab} + \frac{1}{2}\right)\Theta_{ab} + \frac{\pi}{4}\right), \tag{55}$$

where Vol is the volume of a geometric tetrahedron with edge lengths $\lambda k_{ab} + \frac{1}{2}$ and $\Theta_{ab}$ are the dihedral angles. Note that the formula scales as $\lambda^{-3/2}$ due to the volume term. Currently, our formula for the tetrahedron contains the Regge action but the amplitude, phase term and scaling do not obviously agree with (55). We will first consider the intertwiner normalization, which will be necessary to obtain the correct scaling behaviour and some numerical factors, and then evaluate the Hessian numerically to check the agreement of the remaining terms. The main drawback of the coherent state approach occurs here as it is very difficult to obtain an analytic formula for the determinant of the Hessian matrix.

*5.1.1. Intertwiner normalisation.* For the $6j$ symbol, the three valent intertwiners are normalized by dividing by the square root of the theta spin network. The coherent intertwiners that we replaced these with, however, are not normalized. The normalization of these intertwiners for the coherent tetrahedron was studied in [13] so we will briefly summarize the results for the case of the coherent triangle.

The normalization of the coherent intertwiner is given in terms of the three edge vectors of the triangle $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ by the Hermitian inner product

$$\begin{aligned} f_\Delta(\mathbf{n}_i, k_i) &= \int_{SU(2)} \mathrm{d}X \prod_{i=1}^{3} \langle \mathbf{n}_i, k_i | X | \mathbf{n}_i, k_i \rangle \\ &= \int_{SU(2)} \mathrm{d}X \exp S_\Delta, \end{aligned} \tag{56}$$

where

$$S_\Delta = \sum_{i=1}^{3} 2k_i \ln \langle \mathbf{n}_i | X | \mathbf{n}_i \rangle. \tag{57}$$

This integral can be calculated exactly using [16, 17], the result being

$$f_\Delta = \frac{(1-\mathbf{n}_1.\mathbf{n}_2)^p(1-\mathbf{n}_1.\mathbf{n}_2)^q(1-\mathbf{n}_1.\mathbf{n}_2)^r(p+q)!(q+r)!(p+r)!}{2^{p+q+r}(p+q+r+1)!p!q!r!}, \tag{58}$$

where $p = k_1 + k_2 - k_3$, $q = k_2 + k_3 - k_1$ and $r = k_1 + k_3 - k_2$.

The asymptotics of this intertwiner normalization can also be found using the stationary phase [13]. The stationarity of the action $S_\Delta$ gives the closure condition and the action evaluated on the critical points $\pm I$ gives zero:

$$\begin{aligned} f_\Delta(\mathbf{n}_i, \lambda k_i) &\sim \left(\frac{2\pi}{\lambda}\right)^{3/2} \frac{2}{(4\pi)^2} \frac{1}{\sqrt{\det H_\Delta}} \\ &= \frac{1}{\sqrt{2^3\pi\lambda^3 \det H_\Delta}}. \end{aligned} \tag{59}$$

The additional factor 2 comes from the fact that both $I$ and $-I$ are critical points that give the same contribution to the action. $H_\Delta$ is the Hessian matrix of the action which is given by

$$\begin{aligned} H_\Delta^{ij} &= \frac{\partial^2 S_\Delta}{\partial X^i \partial X^j} \\ &= \frac{1}{2}\sum_l k_l\left(\delta^{ij} - n_l^i n_l^j\right). \end{aligned} \tag{60}$$

We can now normalize our formula such that it agrees with the standard normalization by dividing by a factor $(f_{\Delta_a})^{1/2}$ for each triangle $a$.

*5.1.2. Numerical calculations.* With the intertwiner normalizations included in the asymptotic formula, we obtain

$$\begin{aligned} \begin{Bmatrix} \lambda k_{12} & \lambda k_{13} & \lambda k_{14} \\ \lambda k_{23} & \lambda k_{24} & \lambda k_{34} \end{Bmatrix} &= \frac{\mathcal{Z}_{\mathrm{PR}}(\Psi_\lambda, \sigma)}{\prod_{p=1}^4 \sqrt{f_{\Delta_p}}} = \left(\frac{2\pi}{\lambda}\right)^{9/2} \frac{2^4}{((4\pi)^2)^3\sqrt{|\det H|}} \frac{1}{\prod_{p=1}^4 \sqrt{f_{\Delta_p}}} \\ &\quad\times \cos\left(\sum_{a<b} \lambda k_{ab}\Theta_{ab} - \frac{1}{2}\mathrm{Arg}(\det H)\right). \end{aligned} \tag{61}$$

Note that we have the correct scaling behaviour once the additional scaling factors from the intertwiners are included. The normalization terms are real so they do not contribute any additional phase.

The formula for the equilateral tetrahedron with both the exact and approximate intertwiner normalization was compared to the $6j$ symbol and the Ponzano–Regge asymptotic formula using Mathematica in figure 4. We see that our formula differs from the Ponzano–Regge formula for low spins. The only point where our formula differs from Ponzano–Regge is in the fact that the Ponzano–Regge asymptotics are given in terms of the dihedral angles and volume of the tetrahedron with edge lengths $\lambda k_{ab} + \frac{1}{2}$. Therefore, the dihedral angles and volume change nontrivially with $\lambda$. A stationary phase approximation extracts only the scaling behaviour with respect to lambda in the asymptotic regime and cannot register this type of low spin behaviour. This agrees as well as the PR formula for larger spins; however, the agreement for very low values is not as good—figure 4.

## 6. Example: Steffen's flexible polyhedron

Here we discuss an example for which the second part of theorem 1 is relevant, that is, we describe a set of boundary data that admits a flexible immersion. This particular example

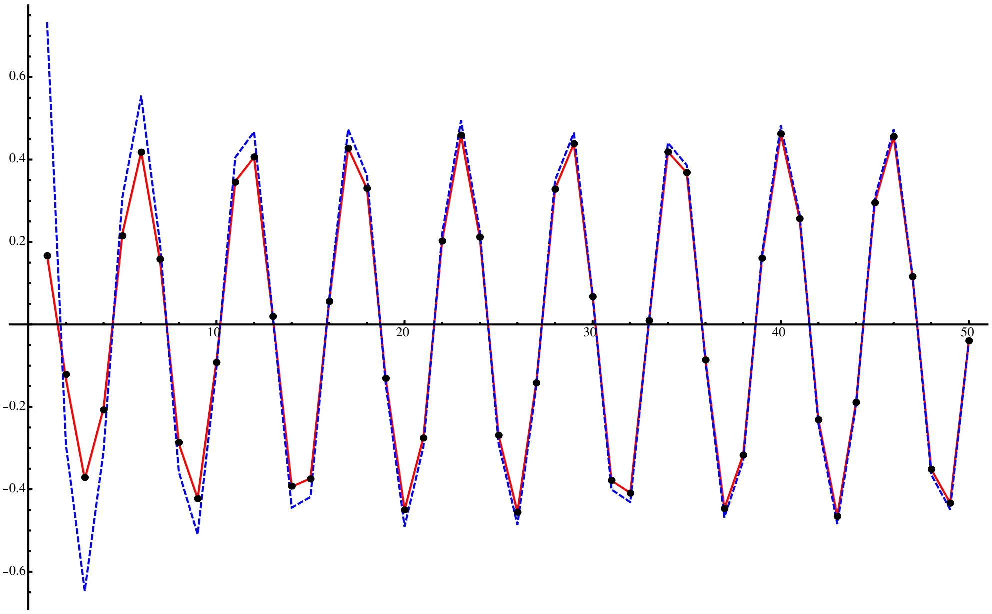


**Figure 4.** Comparison of the $6j$ symbol (dots), the PR formula (solid (red) line) and equation (61) ((blue) dashed line) against the spins $\lambda j$ for the $6j$ symbol with all spins equal. The scaling factor $\lambda^{-3/2}$ has been removed to make the comparison easier at low spins.

(This figure is in colour only in the electronic version)

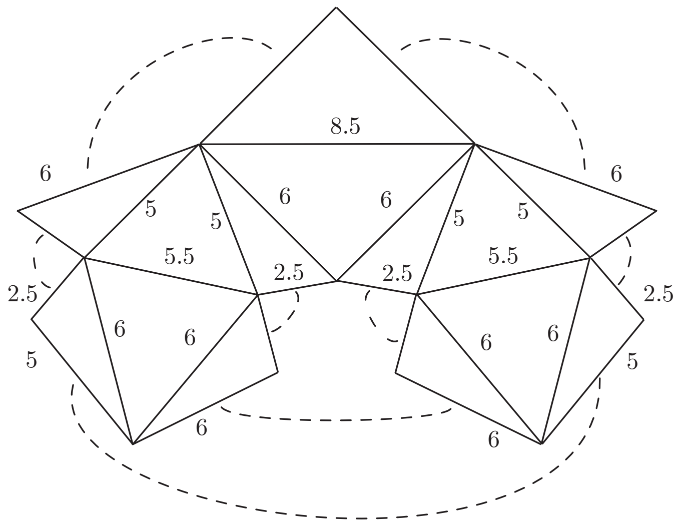


**Figure 5.** A net showing a set of boundary data that reconstructs Steffen's flexible polyhedron.

is taken from a flexible polyhedron with half-integer edge lengths consisting of 14 boundary triangles which was found by Steffen [22]. A net for constructing this polyhedron is given in figure 5 and the corresponding spin network in figure 6.

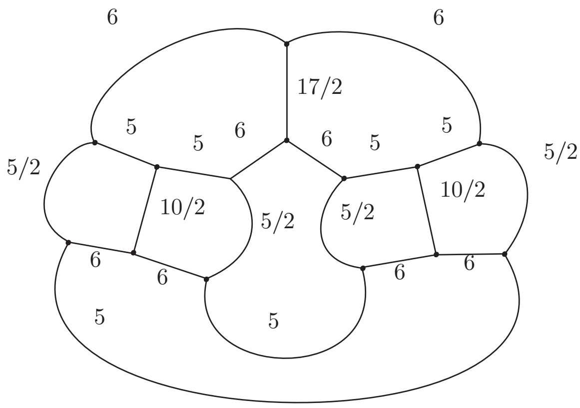


**Figure 6.** The spin network corresponding to Steffen's flexible polyhedron.

Since Steffen's polyhedron admits a flex in one direction, we know that the flexifold is at least one dimensional. As a polyhedron, it is not allowed to self-intersect but there may be other immersions with flexibility in more than one dimension. Applying the asymptotic formula with the same intertwiner normalization as the tetrahedron in section 5, we would expect the scaling to be $\lambda^{-17/2}$.

## 7. Discussion and conclusions

### *7.1. Rigidity of cut immersions*

As discussed, the asymptotic formula produces a sum over all possible immersions of the boundary data in $\mathbb{R}^3$, including flexible ones. These flexible immersions scale differently and thus dominate the rigid immersions. The question of whether a particular polyhedron is rigid is a difficult long-standing problem in mathematics. A classic result is that convex polyhedra are in fact rigid; however, this does not extend to immersions and non-convex polyhedra where counter examples, like Steffen's polyhedron discussed above, are known.

If the boundary data are topologically $\mathcal{S}^2$, then a theorem by Steinitz [23] applies that states that any simplicial complex with underlying space homeomorphic to a 2-sphere admits a simplexwise linear embedding into $\mathbb{R}^3$ whose image is strictly convex. This embedding will indeed be rigid and we can conclude that for the ball $\mathfrak{I}_r$ will always be non-empty.

To our knowledge the only more general results on rigidity of immersions are those giving conditions on bar frameworks, that is a graph immersed in $\mathbb{R}^d$, to be generically rigid. A bar framework is considered generic if the coordinates of the vertices are algebraically independent over the rationals, that is, there is no polynomial with rational coefficients that has these coordinates as roots. A graph is generically rigid if all its generic frameworks are. A set of sufficient and necessary conditions for a graph to be generically rigid are known [24, 25]. Unfortunately this does of course not cover our case with half-integer edge lengths.

Concerning the rigidity of cut immersions, which can be seen as bar frameworks with additional constraints, nothing is known.

### *7.2. Surface immersions versus interior immersions*

With the asymptotic analysis performed above, we explicitly obtain a sum over immersions of the boundary data weighted by the cosine of the Regge action for the immersed surface.

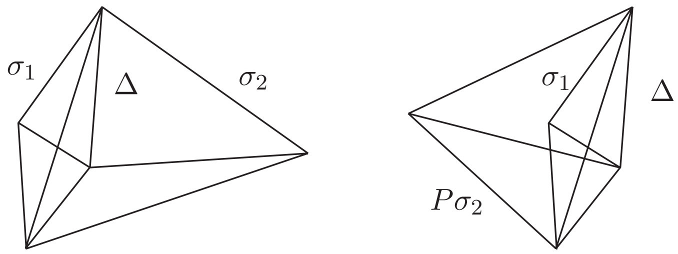


**Figure 7.** Two different possible immersions of the boundary data for two tetrahedra $\sigma_1, \sigma_2$ glued on a common triangle $\Delta$.

Previously, asymptotics of the Ponzano–Regge model for larger triangulations could only be considered by taking the product of the asymptotic formula for each $6j$ symbol. We now illustrate schematically that, in a simple example, this is in fact equivalent to the asymptotic formula above.

We will consider the case of two tetrahedra $\sigma_1, \sigma_2$ glued along a common triangle $\Delta$ and use the boundary normalization that agrees with the $6j$ symbol. The partition function then reads

$$\mathcal{Z}_{\mathrm{PR}}(\Psi_\lambda, \sigma_1 \cup_\Delta \sigma_2) = \begin{Bmatrix} \lambda k_1 & \lambda k_2 & \lambda k_3 \\ \lambda k_4 & \lambda k_5 & \lambda k_6 \end{Bmatrix} \begin{Bmatrix} \lambda k_1 & \lambda k_2 & \lambda k_3 \\ \lambda k_7 & \lambda k_8 & \lambda k_9 \end{Bmatrix}. \tag{62}$$

We write the asymptotic formula for the $6j$ in terms of the Regge action $S_\sigma$ for a tetrahedron $\sigma$

$$\begin{Bmatrix} \lambda k_1 & \lambda k_2 & \lambda k_3 \\ \lambda k_4 & \lambda k_5 & \lambda k_6 \end{Bmatrix} \simeq N\left(\exp(\mathrm{i}\lambda S_\sigma) + \exp(-\mathrm{i}\lambda S_\sigma)\right), \tag{63}$$

where several of the factors have been absorbed into the amplitude $N$. Asymptotically, this gives

$$\begin{aligned} \mathcal{Z}_{\mathrm{PR}}(\Psi_\lambda, \sigma_1 \cup_\Delta \sigma_2) &= N_1 N_2 \left(\exp\left(\mathrm{i}\lambda\left(S_{\sigma_1} + S_{\sigma_2}\right)\right) + \exp\left(-\mathrm{i}\lambda\left(S_{\sigma_1} + S_{\sigma_2}\right)\right)\right) \\ &\quad + N_1 N_2 \left(\exp\left(\mathrm{i}\lambda\left(S_{\sigma_1} - S_{\sigma_2}\right)\right) + \exp\left(-\mathrm{i}\lambda\left(S_{\sigma_1} - S_{\sigma_2}\right)\right)\right) \\ &= N_1 N_2 \exp\left(\sum_{e\subset\Delta} k_e \pi\right) \left(\cos\left(\lambda(S_{\sigma_1\cup_\Delta\sigma_2}\right) + \cos\left(\lambda(S_{\sigma_1\cup_\Delta P\sigma_2}\right)\right), \end{aligned} \tag{64}$$

where $P\sigma$ is the parity-related tetrahedron and we have used the fact that the Regge action for two tetrahedra becomes

$$S_{\sigma_1} + S_{\sigma_2} = S_{\sigma_1\cup_\Delta\sigma_2} + \sum_{e\subset\Delta} k_e \pi. \tag{65}$$

Thus the formula gives a sum over the two different ways of immersing the boundary triangles in $\mathbb{R}^3$, see figure 7.

### *7.3. Boundary states*

We also note that it is possible to select a particular immersion in the sum by choosing a boundary state peaked around a particular set of dihedral angles, see for example [14, 26, 27]. This boundary state also selects one overall orientation of the immersion which removes the parity-related term in the asymptotic formula. For non-rigid immersions, the boundary state would also have the ability to select a particular configuration of the immersed surface which would stop these immersions dominating the integral.

A possible problem with the boundary state is that while it selects an orientation for the boundary, it was not clear if the orientations of the interior tetrahedra behaved consistently. This was considered in [28] and our result also suggests that these do not cause a problem as the asymptotic formula does not register these orientations.

### *7.4. Conclusions*

In this paper we addressed the problem of asymptotics of larger triangulations for the Ponzano–Regge model, by reformulating the partition function as a spin network on the boundary and then rewriting this amplitude using $SU(2)$ coherent states. While this particular feature will not be available for non-topological theories one could expect that in general boundary data will be not suppressed if it can be continued to a solution of the equations of motion on the interior. The asymptotic formula contains a sum over immersions of the boundary data weighted by the cosine of the Regge action. Interestingly, Ponzano and Regge point out in [1] that the different possible immersions corresponding to 3-$nj$ symbols should contribute to the asymptotics but did not obtain a concrete formula. The presented work opens up the possibility of doing an exhaustive analysis of the classical limit of the Ponzano–Regge model including correlation functions on the boundary. As such it can serve as a toy model and proof of concept for conceptual issues likely to arise in all background-independent theories.

Of further interest would be to consider in more detail how the asymptotics obtained here can be obtained from the ‘product of cosines’ picture. In particular to shed light on the issue of causality and orientation in spin foam models.

Interestingly, and unexpectedly, we found that spin networks contain some information about the rigidity properties of surfaces. The scaling properties of a spin network correspond to the maximum dimension of flexibility if the geometry to which it corresponds has any non-rigid immersions.

## Acknowledgments

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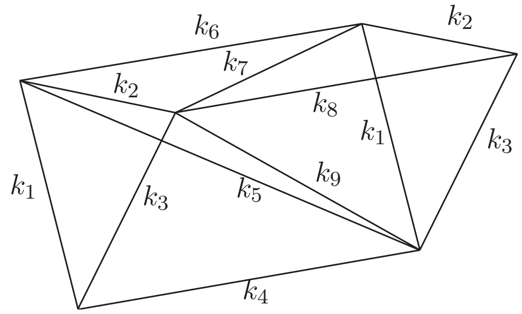
## Appendix A. Example of the Ponzano–Regge amplitude as a spin network on the boundary of the solid torus

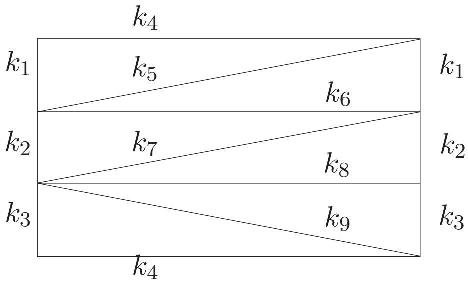
Here we give a simple example of lemma 2 on the solid torus $\mathbb{T}$. A non-tardis (degenerate) triangulation of the solid torus with three tetrahedra is given by



The two triangles with edges $k_1, k_2, k_3$ are identified. The Ponzano–Regge amplitude is given by

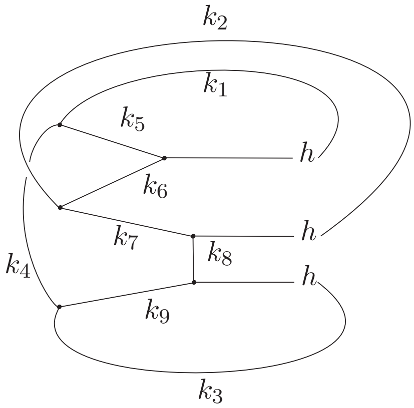
$$\mathcal{Z}_{\mathrm{PR}}(\Psi, \mathbb{T}) = \begin{Bmatrix} k_1 & k_2 & k_3 \\ k_8 & k_9 & k_7 \end{Bmatrix} \begin{Bmatrix} k_1 & k_2 & k_3 \\ k_9 & k_4 & k_5 \end{Bmatrix} \begin{Bmatrix} k_1 & k_5 & k_6 \\ k_2 & k_7 & k_9 \end{Bmatrix}. \tag{A.1}$$

We choose the cutting disc $\mathbb{D}$ to be the triangle $k_1, k_2, k_3$ and perform the cut that reduces $\mathbb{T}$ to the 3-ball. A net for constructing the triangulation on the boundary is given by




The Ponzano–Regge amplitude can be expressed as the following spin network evaluation on the boundary, with a group integral inserted on each of the dual edges that cross $\mathbb{D}$

$$\mathcal{Z}_{PR}(\Psi, \mathbb{T}) = \int_{\mathrm{SU}(2)} dh$$




Expressing this spin network in terms of $6j$ symbols gives equation (A.1).

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